SOLUTIONS FOR PRACTICE PAPER - 3

SECTION – A

I. Very Short Answer Questions.

1. If A = $\left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$ and f : A → B is a surjection defined by f(x) = cos x then find B.

Sol. ∵ f : A → B is surjection and f(x) = cos x.

Now B = f(A)

$$A = \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$$

$$f(0) = \cos(0) = 1$$

$$f\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

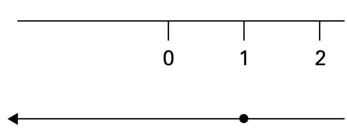
$$f\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\therefore B = f(A) = \left\{1, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}, 0\right\}$$

2. Find the domain of the real-valued function f(x) = $\frac{1}{\log(2-x)}$.

Sol. $f(x) = \frac{1}{\log(2-x)}$



$2 - x > 0$ and $2 - x \neq 1$

$x < 2$ and $x \neq 1$

$\therefore x \in (-\infty, 1) \cup (1, 2)$

3. A certain bookshop has 10 dozen Chemistry books, 8 dozen Physics books, 10 dozen Economics books. Their selling prices are Rs. 80, Rs. 60 and Rs. 40 each respectively. Find the total amount the bookshop will receive by selling all the books, using matrix algebra.

Sol. Number of books A = $\begin{bmatrix} \text{Che} & \text{Phy} & \text{Eco} \\ 10\times 12 & 8\times 12 & 10\times 12 \\ =120 & =96 & =120 \end{bmatrix}$

Selling price in rupees B = $\begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix} \begin{matrix} \text{Che} \\ \text{Phy} \\ \text{Eco} \end{matrix}$

Total value of the books in the shop

$$AB = [120 \ \ 96 \ \ 120] \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix}$$

$$= [120 \times 80 + 96 \times 60 + 120 \times 40]$$

$$= [9600 - 5659 + 4800]$$

$$= [20160] \text{ in rupees}$$

4. If A = $\begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$, then find A + A^2 and AA'.

Sol. $A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$

$$\Rightarrow A + A' = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -9 \\ -9 & 6 \end{bmatrix}$$

$$A.\,A' = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+16 & -10-12 \\ -10-12 & 25+9 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & -22 \\ -22 & 34 \end{bmatrix}$$

5. Show that the points whose position vectors are $-2\bar{a}+3\bar{b}+5\bar{c}$, $\bar{a}+2\bar{b}+3\bar{c}$, $7\bar{a}-\bar{c}$ are collinear when $\bar{a},\bar{b},\bar{c}$ are non-coplanar vectors.

Sol. Let O be the origin and let P, Q, R be the P.V.S. of the given points

$\overline{OP} = -2\bar{a}+3\bar{b}+5\bar{c}$

$\overline{OQ} = \bar{a}+2\bar{b}+3\bar{c}$

$\overline{OR} = 7\bar{a}-\bar{c}$

Then $\overline{PQ} = \overline{OQ}-\overline{OP} = 3\bar{a}-\bar{b}-2\bar{c}$

$\overline{QR} = \overline{OR}-\overline{OQ} = 6\bar{a}-2\bar{b}-4\bar{c} = 2(3\bar{a}-\bar{b}-2\bar{c}) = 2\overline{PQ}$

$\therefore\ \overline{QR} = 2\overline{PQ}$

$\Rightarrow$ The points P, Q, R are collinear.

6. Let $\bar{a}=2\bar{i}+4\bar{j}-5\bar{k}$, $\bar{b}=\bar{i}+\bar{j}+\bar{k}$ and $\bar{c}=\bar{j}+2\bar{k}$. Find unit vector in the opposite direction of $\bar{a}+\bar{b}+\bar{c}$.

Sol. Here $\bar{a}+\bar{b}+\bar{c} = (2\bar{i}+4\bar{j}-5\bar{k}) + (\bar{i}+\bar{j}+\bar{k}) + (\bar{j}+2\bar{k})$

$\Rightarrow\ \bar{a}+\bar{b}+\bar{c} = 3\bar{i}+6\bar{j}-2\bar{k}$

$|\bar{a}+\bar{b}+\bar{c}| = |3\bar{i}+6\bar{j}-2\bar{k}|$

$= \sqrt{(3)^2+(6)^2+(-2)^2}$

$= \sqrt{9+36+4} = \sqrt{49} = 7$

$\therefore$ Unit vector in the opposite direction of

$\bar{a}+\bar{b}+\bar{c}$ is $= -\dfrac{(\bar{a}+\bar{b}+\bar{c})}{|\bar{a}+\bar{b}+\bar{c}|} = -\dfrac{(3\bar{i}+6\bar{j}-2\bar{k})}{7}$

7. If $\bar{a}=\bar{i}+2\bar{j}-3\bar{k}$ and $\bar{b}=3\bar{i}-2\bar{j}+2\bar{k}$ then show that $\bar{a}+\bar{b}$ and $\bar{a}-\bar{b}$ are perpendicular to each other.

Sol. Given that $\bar{a}=\bar{i}+2\bar{j}-3\bar{k}$ and $\bar{b}=3\bar{i}-\bar{j}+2\bar{k}$

Then $\bar{a}+\bar{b} = (\bar{i}+2\bar{j}-3\bar{k}) + (3\bar{i}-\bar{j}+2\bar{k})$

$\therefore\ \bar{a}+\bar{b} = 4\bar{i}+\bar{j}-\bar{k}$

Similarly $(\bar{a}-\bar{b}) = (\bar{i}+2\bar{j}-3\bar{k}) - (3\bar{i}-\bar{j}+2\bar{k})$

$\therefore\ (\bar{a}-\bar{b}) = -2\bar{i}+3\bar{j}-5\bar{k}$

Now $(\bar{a}+\bar{b}).(\bar{a}-\bar{b}) = (4\bar{i}+\bar{j}-\bar{k}).(-2\bar{i}+3\bar{j}-5\bar{k})$

$= -8 + 3 + 5 = 0$

$\because (\bar{a}+\bar{b}).(\bar{a}-\bar{b}) = 0$

$\Rightarrow$ The vectors $\bar{a}+\bar{b}$ and $\bar{a}-\bar{b}$ are mutually perpendicular.

8. Prove that $\dfrac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ}$ = cot 36°.

Sol. L.H.S. $= \dfrac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ}$

Dividing Nr and Dr by cos 9°

$= \dfrac{1+\tan 9^\circ}{1-\tan 9^\circ}$

$= \tan(45^\circ + 9^\circ) \qquad \left(\because \tan(45^\circ + A) = \dfrac{1+\tan A}{1-\tan A}\right)$

$= \tan(54^\circ)$

$= \tan(90^\circ - 36^\circ)$

$= \cot 36^\circ$

$= \cot 36^\circ =$ RHS.

9. Find the period of the function defined by f(x) = tan (x + 4x + 9x + ... + n^2x).

Sol. $f(x) = \tan(x + 4x + 9x + \ldots\ldots + n^2x)$

$f(x) = \tan(1 + 2^2 + 3^2 + \ldots\ldots + n^2)\,x$

$= \tan\left(\dfrac{n(n+1)(2n+1)}{6}\right)x$

$\therefore$ The period of f(x) is $\dfrac{6\pi}{n(n+1)(2n+1)}$

10. If sinh x = 3 then show that x = $\log_e(3+\sqrt{10})$.

Sol. Given sin h(x) = 3

$\Rightarrow x = \sinh^{-1}(3)$

$= \log_e\left(3+\sqrt{3^2+1}\right) = \log_e(3+\sqrt{10})$

$\left[\because \sinh^{-1} x = \log_e\left(x+\sqrt{x^2+1}\right) \forall\, x \in R\right]$

SECTION – B

II. Short Answer Questions.

11. Show that $\begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix}$ **= (a – b) (b – c) (c – a).**

Sol. L.H.S. $\begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix} = \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \end{matrix}$

$$= \begin{vmatrix} bc & b+c & 1 \\ c(a-b) & a-b & 0 \\ a(b-c) & b-c & 0 \end{vmatrix}$$

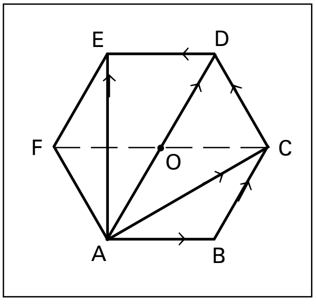
$$= (a-b)\,(b-c) \begin{vmatrix} bc & b+c & 1 \\ c & 1 & 0 \\ a & 1 & 0 \end{vmatrix}$$

$$= (a-b)\,(b-c)\,(c-a)$$

12. Let ABCDEF be regular hexagon with centre 'O'. Show that $\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 3\overline{AD} = 6\overline{AO}$ **.**

Sol. Let ABCDEF be a regular hexagon with centre O.

Then AB = BC = CD = DE = EF = FA



Now $\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF}$

$= (\vec{AC} + \vec{AF}) + \vec{AD} + (\vec{AE} + \vec{AB})$

$= (\vec{AC} + \vec{CD}) + \vec{AD} + (\vec{AE} + \vec{ED})$

$= \vec{AD} + \vec{AD} + \vec{AD} = 3\vec{AD}$

$= 6(\overline{AO}), \because \overline{AD} = 2\overline{AO}$

13. If $\bar{a} = \bar{i} - 2\bar{j} - 3\bar{k}$, $\bar{b} = 2\bar{i} + \bar{j} - \bar{k}$ and $\bar{c} = \bar{i} + 3\bar{j} - 2\bar{k}$ find $\bar{a}\ \ (\bar{b}\ \ c)^{-}$.

Sol. Given that $\bar{a} = \bar{i} - 2\bar{j} - 3\bar{k}$

$\bar{b} = 2\bar{i} + \bar{j} - \bar{k}$ and

$\bar{c} = \bar{i} + 3\bar{j} - 2\bar{k}$

Now $\bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & -1 \\ 1 & 3 & -2 \end{vmatrix}$

$= \bar{i}(-2 + 3) - \bar{j}(-4 + 1) + \bar{k}(6 - 1)$

$= \bar{i} + 3\bar{j} + 5\bar{k}$

Similarly $\bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & -3 \\ 1 & 3 & 5 \end{vmatrix}$

$= \bar{i}(-10 + 9) - \bar{j}(5 + 3) + \bar{k}(3 + 2)$

$= -\bar{i} - 8\bar{j} + 5\bar{k}$

$\therefore \bar{a} \times (\bar{b} \times \bar{c}) = -\bar{i} - 8\bar{j} + 5\bar{k}$

14. If A is not an integral multiple of $\frac{\pi}{2}$, prove that

(i) tan A + cot A = 2 cosec 2A

(ii) cot A – tan A = 2 cot 2A

Sol. (i) $\tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\cos A \sin A}$

$= \frac{1}{\sin A \cos A}$

$= \frac{2}{\sin 2A} = 2 \operatorname{cosec} 2A$

(ii) $\cot A - \tan A = \dfrac{1}{\tan A} - \tan A = \dfrac{1-\tan^2 A}{\tan A}$

$$= 2\left(\frac{1-\tan^2 A}{2\tan A}\right) = \frac{2}{\tan 2A}$$

$$= 2\cot 2A$$

15. Solve : $2\cos^2\theta - \sqrt{3}\sin\theta + 1 = 0$.

Sol. Given that

$2\cos^2\theta - \sqrt{3}\sin\theta + 1 = 0$

$\Rightarrow 2(1 - \sin^2\theta) - \sqrt{3}\sin\theta + 1 = 0$

$\Rightarrow 2 - 2\sin^2\theta - \sqrt{3}\sin\theta + 1 = 0$

$\Rightarrow 2\sin^2\theta + \sqrt{3}\sin\theta - 3 = 0$

$\therefore \sin\theta = \dfrac{-\sqrt{3} + \sqrt{3 - 4(2)(-3)}}{2(2)}$

$= \dfrac{-\sqrt{3} \pm \sqrt{27}}{4}$

$= \dfrac{-\sqrt{3} + 3\sqrt{3}}{4}, \dfrac{-\sqrt{3} - 3\sqrt{3}}{4}$

$= \dfrac{\sqrt{3}}{2}, -\sqrt{3}$

$\because \sin\theta \in [-1,1]$, $\sin\theta = -\sqrt{3}$ is not admisible

$\therefore \sin\theta = \dfrac{\sqrt{3}}{2} = \sin\dfrac{\pi}{3}$

Hence the principle value of $\theta = \dfrac{\pi}{3}$ and the general solution for

$\theta = n\pi + (-1)^n\dfrac{\pi}{3}$; $n \in z$.

16. Prove that $\cos\left(2\tan^{-1}\dfrac{1}{7}\right) = \sin\left(4\tan^{-1}\dfrac{1}{3}\right)$.

Sol. Let $\text{Tan}^{-1}\left(\dfrac{1}{7}\right) = \alpha$ and $\text{Tan}^{-1}\left(\dfrac{1}{3}\right) = \beta$

$\tan\alpha = \frac{1}{7}$ and $\tan\beta = \frac{1}{3}$

L.H.S. $\cos\left(2\tan^{-1}\frac{1}{7}\right) = \cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$

$$= \frac{1-\frac{1}{49}}{1+\frac{1}{49}} = \frac{48}{49} = \frac{24}{25}$$

$\because \tan\beta = \frac{1}{3} \Rightarrow \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2\left(\frac{1}{3}\right)}{1-\frac{1}{9}}$

$$= \frac{2}{3}\times\frac{9}{8} = \frac{3}{4}$$

R.H.S. $= \sin\left(4\tan^{-1}\frac{1}{3}\right)$

$$= \sin 4\beta = \sin(2\times 2\beta) = \frac{2\tan 2\beta}{1+\tan^2(2\beta)} = \frac{2\left(\frac{3}{4}\right)}{1+\frac{9}{16}}$$

$$= \frac{6}{4}\times\frac{16}{25} = \frac{24}{25}$$

$\therefore$ L.H.S. = R.H.S.

17. In a ΔABC prove that $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$.

Sol. $\left(\frac{b-c}{b+c}\right)\cot\frac{A}{2} = \frac{2R\sin B - 2R\sin C}{2R\sin B + 2R\sin C}.\cot\frac{A}{2}$

$$= \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$= \frac{\cancel{2}\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{\cancel{2}\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}.\ \cot\frac{A}{2}$$

$$= \cot\left(\frac{B+C}{2}\right)\tan\left(\frac{B-C}{2}\right).\cot\frac{A}{2}$$

$$= \cot\left(90-\frac{A}{2}\right)\tan\left(\frac{B-C}{2}\right)\cot\frac{A}{2}$$

$$= \tan\frac{A}{2}\tan\left(\frac{B-C}{2}\right).\cot\frac{A}{2}$$

$$= \tan\left(\frac{B-C}{2}\right)$$

SECTION – C

III. Long Answer Questions.

18. Let $f : A \to B$, $g : B \to C$ be bijections. Then prove that $(gof)^{-1} = f^{-1}og^{-1}$.

Sol. $f : A \to B$, $g : B \to C$ are bijections

$\Rightarrow gof : A \to C$ is a bijection

Also $g^{-1} : C \to B$ and $f^{-1} : B \to A$ are bijections

$\Rightarrow f^{-1}og^{-1} : C \to A$ is a bijection.

Let c be any element of C.

Then $\exists$ an element $b \in B$ such that $g(b) = c$

$\Rightarrow b = g^{-1}(c)$

Also $\exists$ an element $a \in A$ such that $f(a) = b$

$\Rightarrow a = f^{-1}(b)$

Now $(gof)(a) = g(f(a)) = g(b) = c$

$\Rightarrow a = (gof)^{-1}(c) \Rightarrow (gof)^{-1}(c) = a$ ——— (1)

Also $(f^{-1}og^{-1})(c) = f^{-1}(g^{-1}(c)) = f^{-1}(b) = a$ ——— (2)

$\therefore$ From (1) and (2) ;

$(gof)^{-1}(c) = (f^{-1}og^{-1})(c)$

$\Rightarrow (gof)^{-1} = f^{-1}og^{-1}$.

19. By using mathematical induction show that $\forall$ n $\in$ N,

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10} + \text{....... upto n terms} = \frac{n}{3n+1}.$$

Sol. Here 1, 4, 7 .... are in A.P.

$a = 1, d = 3$

Its n^{th} term $= a + (n-1)d$

$= 1 + (n-1)3$

$= 3n - 2$

Similarly 4, 7, 10, ... are in A.P.

Its n^{th} term $= 4 + (n-1)3$

$= 3n + 1$

$a = 4, d = 3$

$\therefore$ n^{th} term of the given series $= \dfrac{1}{(3n-2)(3n+1)}$

Let S(n) be the given statement

(i.e.,) $\dfrac{1}{1.4}+\dfrac{1}{4.7}+..........+\dfrac{1}{(3n-2)(3n+1)}=\dfrac{n}{3n+1}$

When n = 1, LHS $= \dfrac{1}{1.4} = \dfrac{1}{4}$

RHS $= \dfrac{1}{3(1)+1}=\dfrac{1}{4}$

$\therefore$ LHS = RHS for n = 1

$\therefore$ S(1) is true

Let us suppose that S(k) is true

(i.e.,) $\dfrac{1}{1.4}+\dfrac{1}{4.7}+..........+\dfrac{1}{(3k-2)(3k+1)}=\dfrac{k}{3k+1}$

Add $\dfrac{1}{[3(k+1)-2][3(k+1)+1]}=\dfrac{1}{(3k+1)(3k+4)}$ on both sides

$$\frac{1}{1.4}+\frac{1}{4.7}+.........+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k(3k+4)+1}{(3k+1)(3k+4)}$$

$$=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$=\frac{(k+1)(3k+1)}{(3k+1)(3k+4)}$$

$$=\frac{k+1}{3(k+1)+1}$$

$\therefore$ S (k + 1) is true.

Hence by the principle of Mathematical Induction S(h) is true $\forall$ n $\in$ N.

(i.e.,) $\frac{1}{1.4}+\frac{1}{4.7}$ + ... to n terms = $\frac{n}{3n+1}$

20. If A = $\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$ then find $(A')^{-1}$.

Sol. det A = 1(–1 – 8) + 2(0 + 8) + 3(0 – 2)

= –9 + 16 – 6

= 1

Cofactors of A = $\begin{bmatrix} -9 & -8 & -2 \\ +8 & 7 & +2 \\ -5 & -4 & -1 \end{bmatrix}$

Adj A = $\begin{bmatrix} -9 & -8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$

$$\therefore \boxed{A^{-1} = \frac{1}{\det A}\,\text{Adj } A}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} -9 & +8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$$

$$(A^{-1})' = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

We know that $(A^{-1})' = (A^T)'$

$$\therefore (A^T)^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

21. Solve the following equations by Gauss-Jordan method $3x + 4y + 5z = 18$, $2x - y + 8z = 13$ and $5x - 2y + 7z = 20$.

Sol. Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & 2 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$

The augmented matrix is

$$\begin{bmatrix} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix} \sim R_1 \to R_1 - R_2$$

$$\begin{bmatrix} 1 & 5 & -3 & 5 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix}$$

$$\sim \begin{matrix} R_2 \to R_2 - 2R_3 \\ R_3 \to R_3 - 5R_1 \end{matrix} \begin{bmatrix} 1 & 5 & -3 & 5 \\ 0 & -11 & 14 & 3 \\ 0 & -27 & 22 & -5 \end{bmatrix}$$

$$\sim R_2 \to -5R_2 + 2R_3 \begin{bmatrix} 1 & 5 & -3 & 5 \\ 0 & 1 & -26 & 25 \\ 0 & -27 & 22 & -5 \end{bmatrix}$$

$$\sim \begin{matrix} R_1 \to R_1 - 5R_2 \\ R_3 \to R_3 + 27R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 127 & 130 \\ 0 & 1 & -26 & -25 \\ 0 & 0 & -680 & -680 \end{bmatrix}$$

$$\sim R_3 \to R_3 + (-680) \begin{bmatrix} 1 & 0 & 127 & 135 \\ 0 & 1 & -26 & -25 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{matrix} R_1 \to R_1 - 127\,R_3 \\ R_2 \to R_2 + 26R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Hence the solution is $x = 3$, $y = 1$, $z = 1$.

22. If A = (1, –2, –1), B = (4, 0, –3), C = (1, 2, –1) and D = (2, –4, –5), find the distance between $\overline{AB}$ and $\overline{CD}$.

Sol. Let 'O' be the origin.

$\overline{OA} = \overline{i} - 2\overline{j} - \overline{k}$

$\overline{OB} = 4\overline{i} - 3\overline{k}$

$\overline{OC} = \overline{i} + 2\overline{j} - \overline{k}$

$\overline{OD} = 2\overline{i} - 4\overline{j} - 5\overline{k}$

Then $b = \overline{AB} = \overline{OB} - \overline{OA} = 3\overline{i} + 2\overline{j} - 2\overline{k}$

$d = \overline{CD} = \overline{OD} - \overline{OC} = \overline{i} - 6\overline{j} - 4\overline{k}$

Equation of AB is $\overline{r} = \overline{a} + t\overline{b}$

$\overline{r} = (\overline{i} - 2\overline{j} - \overline{k}) + t(3\overline{i} + 2\overline{j} - 2\overline{k})$

Equation of CD is $\overline{r} = \overline{c} + s\overline{d}$

$\overline{r} = (\overline{i} + 2\overline{j} - \overline{k}) + s(\overline{i} - 6\overline{j} + 4\overline{k})$

$\overline{a} - \overline{c} = (\overline{i} - 2\overline{j} - \overline{k}) - (\overline{i} + 2\overline{j} - 2\overline{k}) = -4\overline{j}$

$$\overline{b} \times \overline{d} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 3 & 2 & -2 \\ 1 & -6 & -4 \end{vmatrix} = \overline{i}(-8 - 12) - \overline{j}(-12 + 2) + \overline{k}(-18 - 2)$$

$$= -20\overline{i} + 10\overline{j} - 20\overline{k}$$

$\left|\bar{b}\times\bar{d}\right| = \sqrt{400+100+400} = \sqrt{900} = 30$

$(\bar{a}-\bar{c}).(\bar{b}\times\bar{d}) = (-4\bar{j}).(-20\bar{i}+10\bar{j}-20\bar{k})$

$= -40$

$\therefore$ Distance between the lines AB and CD

$$= \frac{\left|(\bar{a}-\bar{c}).(\bar{b}\times\bar{d})\right|}{|\bar{b}\times\bar{d}|} = \frac{|-40|}{30} = \frac{4}{3} \text{ units.}$$

23. If A, B, C are angles of a triangle, then prove that

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2} = 1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

Sol. L.H.S. $= \sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}$

$$= \left(\frac{1-\cos A}{2}\right)+\left(\frac{1-\cos B}{2}\right)-\sin^2\frac{c}{2}$$

$$= \frac{1}{2}\left[2-(\cos A+\cos B)\right]-\sin^2\frac{c}{2}$$

$$= \frac{1}{2}\left[2-2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)\right]-\sin^2\frac{c}{2}$$

$$= 1-\cos\left(90^\circ-\frac{c}{2}\right)\cos\left(\frac{A-B}{2}\right)-\sin^2\frac{c}{2}$$

$$= 1-\sin\frac{C}{2}.\cos\left(\frac{A-B}{2}\right)-\sin^2\frac{C}{2}; \qquad \because A+B+C=180^\circ$$

$$= 1-\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\sin\frac{C}{2}\right]$$

$$= 1-\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\sin\left(90^\circ-\frac{A+B}{2}\right)\right]$$

$$= 1-\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right]$$

$$= 1-\sin\frac{C}{2}\left[2\cos\frac{A}{B}\cos\frac{B}{2}\right]$$

$$= 1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

24. In a ΔABC, if a = 13, b = 14, c = 15, find R, r, r_1, r_2 and r_3.

Sol. a = 13, b = 14, c = 15

$$s = \frac{a+b+c}{2} = \frac{13+14+15}{2}$$

$$= \frac{42}{2} = 21$$

$s - a = 21 - 13 = 8$;

$s - b = 21 - 14 = 7$;

$s - c = 21 - 15 = 6$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{21 \times 8 \times 7 \times 6}$$

$$= \sqrt{21 \times 16 \times 21} = 84$$

$$R = \frac{abc}{4\Delta} = \frac{13 \times 14 \times 15}{4 \times 84} = \frac{65}{8} \text{ units}$$

$$r = \frac{\Delta}{s} = \frac{84}{21} = 4 \;;$$

$$r_1 = \frac{\Delta}{s-a} = \frac{84}{8} = \frac{21}{2}$$

$$r_2 = \frac{\Delta}{s-b} = \frac{84}{7} = 12$$

$$r_3 = \frac{\Delta}{s-c} = \frac{84}{6} = 14$$

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