PRACTICE PAPER – 3

SOLUTIONS

SECTION – A

I. Very Short Answer Questions.

1. Find the equation of the circle with centre $C\left(\frac{5}{2}, \frac{-4}{3}\right)$ and radius 6.

Sol. Equation of the circle is

$$\left(x - \frac{5}{2}\right)^2 + \left(y + \frac{4}{3}\right)^2 = 6^2$$

$$\Rightarrow x^2 - 5x + \frac{25}{4} + y^2 + \frac{8}{3}y + \frac{16}{9} = 36$$

$$\Rightarrow x^2 + y^2 - 5x + \frac{8}{3}y + \frac{25}{4} + \frac{16}{9} - 36 = 0$$

Multiplying with 36

$$36x^2 + 36y^2 - 180x + 96y + 225 + 64 - 1296 = 0$$

$$\Rightarrow 36x^2 + 36y^2 - 180x + 96y - 1007 = 0$$

2. Find the equation of the normal at P of the circle S = 0 where P and S are given by P = (3, –4), $S \equiv x^2 + y^2 + x + y - 24$.

Sol. Equation of the normal is

$$(x - x_1)(y_1 + f) - (y - y_1)(x_1 + g) = 0$$

$$(x - 3)\left(-4 + \frac{1}{2}\right) - (y + 4)\left(3 + \frac{1}{2}\right) = 0$$

$$-\frac{7}{2}(x - 3) - \frac{7}{2}(y + 4) = 0$$

$$\Rightarrow (x - 3) + (y + 4) = 0$$

$$x - 3 + y + 4 = 0$$

$$x + y + 1 = 0$$

3. Find the equation of the radical axis of

$x^2 + y^2 - 2x - 4y - 1 = 0$, $x^2 + y^2 - 4x - 6y + 5 = 0$.

Sol. $S - S' = 0$ radical axis

$(x^2 + y^2 - 2x - 4y - 1) - (x^2 + y^2 - 4x - 6y + 5) = 0$

$2x + 2y - 6 = 0$ (or)

$x + y - 3 = 0$

4. A double ordinate of the curve $y^2 = 4ax$ is of length 8a prove that the lines from the vertex to its ends are at right angles.

Sol. Let $P = (at^2, 2at)$ and $P' = (at^2, -2at)$ be the ends of double ordinate PP'. Then

$8a = PP' = \sqrt{0 + (4at)^2} = 4at \Rightarrow t = 2.$

$\therefore P = (4a, 4a), P' = (4a, -4a)$

Slope of $\overline{AP} \times$ slope of $\overline{AP'}$

$= \left(\frac{4a}{4a}\right)\left(-\frac{4a}{4a}\right) = -1$

$\therefore \angle PAP' = \frac{\pi}{2}$.

5. Find the product of lengths of the perpendiculars from any point on the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ to its asymptotes.

Sol. Equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$a^2 = 16, b^2 = 9$

Product of the perpendiculars from any point as the hyperbola to its asymptotes

$= \frac{a^2b^2}{a^2 + b^2} = \frac{16 \times 9}{16 + 9} = \frac{144}{25}$

6. Find $\int \frac{x^8}{1+x^{18}} dx; x \in R$

Sol. $t = x^9 \Rightarrow dt = 9x^8\, dx$

$$\int \frac{x^8\, dx}{1+x^{18}} = \int \frac{x^8}{1+(x^9)^2} dx$$

$$= \frac{1}{9}\int \frac{dt}{1+t^2} = \frac{1}{9}\tan^{-1} t + C = \frac{1}{9}\tan^{-1}(x^9) + C$$

7. Evaluate $\int 2x \sin(x^2+1)\, dx$ on $x \in R$.

Sol. $\int 2x. \sin(x^2+1)\, dx$

$t = x^2 + 1 \Rightarrow dt = 2x\, dx$

$\int 2x. \sin(x^2+1)\, dx = \int \sin t\, dt = -\cos t + C = -\cos(x^2+1) + C$

8. Evaluate $\int_0^2 |1-x|\, dx.$

Sol. $\int_0^1 -(x-1)\, dx + \int_1^2 (x-1)\, dx$

$$= \int_0^1 (-x+1)\, dx + \int_1^2 (x-1)\, dx = \left[\frac{-x^2}{2}+x\right]_0^1 + \left[\frac{x^2}{2}-x\right]_1^2$$

$$= -\frac{1}{2}+1+\left(\frac{4}{2}-2\right)-\left(\frac{1}{2}-1\right) = \frac{1}{2}+\frac{1}{2}=1$$

9. Evaluate $\int_{-\pi/2}^{\pi/2} \sin^3\theta \cos^3\theta\, d\theta.$

Sol. $f(\theta) = \sin^3\theta . \cos^3\theta$

$f(-\theta) = \sin^3(-\theta)\cos^3(-\theta) = -\sin^3\theta\cos^3\theta = -f(\theta)$

$f(\theta)$ is odd

$\therefore \int_{-\pi/2}^{\pi/2} \sin^3\theta.\cos^3\theta\, d\theta = 0$

10. Find the order of the differential equation corresponding to $y = c(x - c)^2$. Where c is an arbitrary constant.

Sol. The given differential equation is

$y = c(x - c)^2$

$$\frac{dy}{dx} = 2c(x - c)$$

$\therefore$ Order of the differential equation is 1.

SECTION – B

II. Short Answer Questions.

11. Show that the equation of the circle whose diameter extremities are $(x_1\ y_1)$ and (x_2, y_2) is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

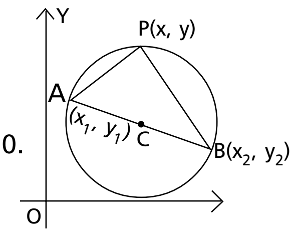
Sol. Let $A = (x_1, y_1)$, $B = (x_2, y_2)$ and C be the centre of the circle

Let P(x, y) be any point on it other than A and B. Join A and B, A and P and also P and B. We know that

$A\hat{P}B = 90°$.

i.e., the lines AP and BP are perpendicular

$\therefore$ (slope of AP) slope of BP) = – 1.



i.e., $\frac{(y - y_1)}{(x - x_1)} \times \frac{(y - y_2)}{(x - x_2)} = -1$

i.e., $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Also clearly A and B satisfy (1).

There fore any point P(x, y) on the circle satisfies equation (1).

Conversely if a point P(x, y) satisfies (1) then $A\hat{P}B = 90°$ and hence P lies on the circle.

Thus (1) is the equation of the required circle.

12. If x + y = 3 is the equation of the chord AB of the circle $x^2 + y^2 - 2x + 4y - 8 = 0$, find the equation of the circle having $\overline{AB}$ as diameter.

Sol. Required equation of circle passing through intersection S = 0 and L = 0 is $S + \lambda L = 0$

$(x^2 + y^2 - 2x + 4y - 8) + \lambda(x + y - 3) = 0$

$(x^2 + y^2 + x(-2 + \lambda) + y(4 + \lambda) - 8 - 3\lambda = 0$ —— (i)

$x^2 + y^2 + 2gx + 2fy + c = 0$ —— (ii)

Comparing (i) and (ii) we get

$g = \frac{(-2+\lambda)}{2}$, $f = \frac{(4+\lambda)}{2}$

Centre lies on x + y = 3

$\therefore -\left(\frac{-2+\lambda}{2}\right) - \left(\frac{4+\lambda}{2}\right) = 3$

$2 - \lambda - 4 - \lambda = 6$

$-2\lambda = 8 \Rightarrow \lambda = -4$

Required equation of circle be

$(x^2 + y^2 - 2x + 4y - 8) - 4(x + y - 3) = 0$

$x^2 + y^2 - 6x + 4 = 0$

13. Find the equation of the ellipse referred to its major and minor axes as the coordinate axes x, y respectively with latus rectum of length 4 and the distance between foci $4\sqrt{2}$.

Sol. Let the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b)

Length of the latus rectum = $\frac{2b^2}{a^2} = 4 \Rightarrow b^2 = 2a$.

Foci are S (ae, 0), S' (– ae, 0)

Distance between the foci = $2ae = 4\sqrt{2}$

$ae = 2\sqrt{2}$

$b^2 = a^2(1 - e^2) = a^2 - (ae)^2$

$2a = a^2 - 8 \Rightarrow a^2 - 2a - 8 = 0$

$(a - 4)(a + 2) = 0$

$a = 4$ or -2

$a > 0 \Rightarrow a = 4$

$b^2 = 2a = 2.\,4 = 8$

Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{16} + \frac{y^2}{8} = 1$$

$$x^2 + 2y^2 = 16.$$

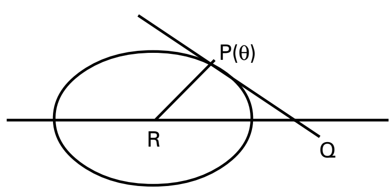
14. The tangent and normal to the ellipse $x^2 + y^2 = 4$ at a point P(0) on it meets the major axis in Q and R respectively. If $0 < \theta < \frac{\pi}{2}$ and QR = 2 then show that $Q = \cos^{-1}\left(\frac{2}{3}\right)$.

Sol. Equation of the ellipse is $x^2 + 4y^2 = 4$

$\frac{x^2}{4} + \frac{y^2}{1} = 1$

Equation of the tangent at P (θ) is

$\frac{x}{2}.\cos\theta + \frac{y}{1}\sin\theta = 1$



Equation of x–axis is y = 0

$\frac{x}{2}\cos\theta = 1 \Rightarrow x = \frac{2}{\cos\theta}$

Co-ordinates of Q are $\left(\frac{2}{\cos\theta}, \theta\right)$

Equation of the normal at

P(θ) is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

$$\frac{2x}{\cos\theta} - \frac{y}{\sin\theta} = 3$$

Substituting y = 0 we get $\frac{2x}{\cos\theta} = 3$

$$x = \frac{3}{2}.\cos\theta$$

Co-ordinates of R are $\left(\frac{3}{2}.\cos\theta, 0\right)$

$$QR = \left(\frac{-3}{2}\cos\theta + \frac{2}{\cos\theta}\right) = \frac{-3\cos^2\theta + 4}{2\cos\theta}$$

Given QR = 2

$$\frac{-3\cos^2\theta + 4}{2\cos\theta} = 2$$

$-3\cos^2\theta + 4 = 4\cos\theta$

$3\cos^2\theta + 4\cos\theta - 4 = 0$

$(3\cos\theta - 2)(\cos\theta + 2) = 0$

$3\cos\theta - 2 = 0$ or $\cos\theta + 2 = 0$

$\cos\theta = \frac{2}{3}$ or $\cos\theta = -2$

$\cos\theta$ always lies between –1 and 1

$$\therefore \quad \cos\theta = \frac{2}{3}$$

$$\text{i.e., } \theta = \cos^{-1}\left(\frac{2}{3}\right).$$

15. A circle cuts the rectangular hyperbola xy = 1, in the point (x_r, y_r) r = 1, 2, 3, 4, prove that $x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4 = 1$.

Sol. Let the circle be $x^2 + y^2 = a^2$.

Since $\left(t, \frac{1}{t}\right)$ $(t \neq 0)$ lies on xy = 1, the points of intersection of the circle and the hyperbola are given by $t^2 + \frac{1}{t^2} = a^2$

$\Rightarrow t^4 - a^2t^2 + 1 = 0$

$\Rightarrow t^4 + 0.t^3 - a^2t^2 + 0.t + 1 = 0.$

If t_1, t_2, t_3 and t_4 are the roots of the above biquadratic, then $t_1t_2t_3t_4 = 1$.

If $(x_r, y_r) = \left(t_r, \frac{1}{t_r}\right)$, r = 1, 2, 3, 4,

then $x_1x_2x_3x_4 = t_1t_2t_3t_4 = 1$,

and $y_1y_2y_3y_4 = \dfrac{1}{t_1t_2t_3t_4} = 1$.

16. Find $\int_0^{\pi/2} \cos^7 x \sin^2 x\, dx$.

Sol. $I = \int_0^{\pi/2} \cos^7 x \,.\, \sin^2 x\, dx$

$\int_0^{\pi/2} \sin^m x . \cos^n x\, dx$

m – even

n – odd

$= \dfrac{n-1}{m+n} \cdot \dfrac{n-3}{m+n-2} \cdots \dfrac{2}{m+3} \cdot \dfrac{1}{m+1}$

$= \dfrac{7-1}{9} \times \dfrac{7-3}{7} \times \dfrac{7-5}{5} \cdot \dfrac{1}{2+1} = \dfrac{6}{9} \cdot \dfrac{4}{7} \cdot \dfrac{2}{5} \cdot \dfrac{1}{3} = \dfrac{16}{315}$

17. Solve $x(x-1)\dfrac{dy}{dx} - (x-2)y = x^3(2x-1)$

Sol. $\dfrac{dy}{dx} - \dfrac{x-2}{x(x-1)} y = \dfrac{x^3(2x-1)}{x(x-1)}$

I.F. $= e^{\int \frac{2-x}{x(x-1)}} dx$

$\dfrac{2-x}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$

$2 - x = A(x-1) + B.x$

$x = 0 \Rightarrow 2 = -A \Rightarrow A = -2$

$x = 1 \Rightarrow 1 = B \quad \Rightarrow B = 1$

$$\frac{2-x}{x(x-1)} = \frac{-2}{x} + \frac{1}{x-1}$$

$$\int \frac{2-x}{x(x-1)}\,dx = -2\int\frac{dx}{x} + \int \frac{dx}{x-1}$$

$$= -2\log x + \log(x-1) = \log\frac{(x-1)}{x^2}$$

$$\text{I.F.} = e^{\log\frac{(x-1)}{x^2}} = \frac{x-1}{x^2}$$

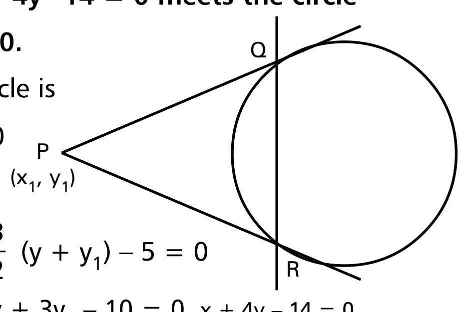
$$y\frac{(x-1)}{x^2} = \int \frac{x^3(2x-1)}{x(x-1)}.\frac{x-1}{x^2}\,dx = \int (2x-1)\,dx = x^2 - x + c$$

Solution is $y(x-1) = x^2(x^2 - x + c)$

SECTION – C

III. Long Answer Questions.

18. Find the coordinates of the point of intersection of tangents at the points where $x + 4y - 14 = 0$ meets the circle $x^2 + y^2 - 2x + 3y - 5 = 0$.



Sol. Equation of the given circle is

$x^2 + y^2 - 2x + 3y - 5 = 0$

polar of $P(x_1, y_1)$ is

$$xx_1 + yy_1 - 1(x + x_1) + \frac{3}{2}(y + y_1) - 5 = 0$$

$2xx_1 + 2yy_1 - 2x - 2x_1 + 3y + 3y_1 - 10 = 0$

$2(x_1 - 1)x + (2y_1 + 3)y - (2x_1 - 3y_1 + 10) = 0$ ——(1)

Equation of QR is $x + 4y - 14 = 0$ ——(2)

Comparing (1) and (2)

$$\frac{2(x_1-1)}{1} = \frac{2y_1+3}{4} = \frac{2x_1-3y_1+10}{14}$$

$$2(x_1-1) = \frac{2y_1+3}{4}$$

$$8x_1 - 8 = 2y_1 + 3$$

$$8x_1 - 2y_1 = 11 \qquad \text{—— (1)}$$

$$2(x_1-1) = \frac{2x_1-3y_1+10}{14}$$

$$28x_1 - 28 = 2x_1 - 3y_1 + 10$$

$$26x_1 + 3y_1 = 38 \qquad \text{—— (2)}$$

$$24x_1 - 6y_1 = 33 \quad (1) \times 3$$

$$52x_1 + 6y_1 = 76 \quad (2) \times 2$$

Adding $76x_1 = 109$

$$\boxed{x_1 = \frac{109}{76}} \qquad \text{—— (3)}$$

From (3) $2y_1 = 8x_1 - 11 = 8 \times \frac{109}{76} - 11$

$$= -\frac{218-209}{19} = \frac{9}{19}$$

$$\boxed{y_1 = \frac{9}{38}}$$

$\therefore$ Co- ordinates of p are $\left(\frac{109}{76}, \frac{9}{38}\right)$

19. Prove that the circles $x^2 + y^2 - 8x - 6y + 21 = 0$ and $x^2 + y^2 - 2y - 15 = 0$ have exactly two common tangents. Also find the point of intersection of those tangents.

Sol. Let C_1, C_2 be the centres and r_1, r_2 be their radii.

Equation of the circles are

$$x^2 + y^2 - 8x - 6y + 21 = 0$$

and $$x^2 + y^2 - 2y - 15 = 0$$

$C_1(4, 3)$, $C_2(0, 1)$

$r_1 = \sqrt{16 + 9 - 21} = 2$, $r_2 = \sqrt{1 + 15} = 4$

$\overline{C_1C_2}^2 = (4 - 0)^2 + (3 - 1)^2 = 16 + 4 = 20$

$C_1C_2 = 2\sqrt{5}$

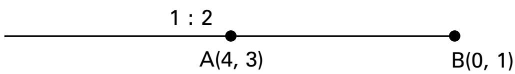
$|r_1 - r_2| = |2 - 4| = 2$, $r_1 + r_2 = 2 + 4 = 6$

$|r_1 - r_2| < C_1C_2 < r_1 + r_2$

Given circles intersect and have exactly two common tangents.

$r_1 : r_2 = 2 : 4 = 1 : 2$

The tangents intersect in external centre of similitude



Co–ordinates of S are

$$\left(\frac{1.0 - 2.4}{1 - 2}, \frac{1.0 - 2.3}{-2}\right) = \left(\frac{-8}{-1}, \frac{-5}{-1}\right) = (8, 5)$$

20. Prove that the area of the triangle formed by the tangents at (x_1, y_1) (x_2, y_2) and (x_3, y_3) to the parabola $y^2 = 4ax$ $(a > 0)$ is $\frac{1}{16a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$ sq units.

Sol. Let $D(x_1, y_1) = (at_1^2, 2at_1)$, $E(x_2, y_2) = (at_2^2, 2at_2)$

and $F(x_3, y_3) = (at_2^3, 2at_3)$

be three points on the parabola

$y^2 = 4ax$ $(a > 0)$.

The equation of the tangents at D, E and F are

$t_1y = x + at_1^2$ —— (1)

$t_2y = x + at_2^2$ —— (2)

$t_3y = x + at_3^2$ —— (3)

$(1) - (2) \Rightarrow (t_1 - t_2)\, y = a(t_1 - t_2)(t_1 + t_2)$

$\Rightarrow y = a(t_1 + t_2)$ substituting in (1)

we get $x = at_1t_2$.

$\therefore$ The point of intersection of the tangents at D and E is say $P(at_1t_2, a(t_1 + t_2))$

Similarly the points of intersection of tangent at E, F and at F, D are $Q(at_2t_3, a(t_2 + t_3)$ and

$R(at_3t_1, a(t_3 + t_1))$ respectively

Area of ΔPQR

$$= \text{Absolute value of } \frac{1}{2}\begin{vmatrix} at_1t_2 & a(t_1+t_2) & 1 \\ at_2t_3 & a(t_2+t_3) & 1 \\ at_1t_3 & a(t_1+t_3) & 1 \end{vmatrix}$$

$$= \text{Absolute value of } \frac{a^2}{2}\begin{vmatrix} t_1t_2 & t_1+t_2 & 1 \\ t_2t_3 & t_2+t_3 & 1 \\ t_1t_3 & t_1+t_3 & 1 \end{vmatrix}$$

$$= \text{Absolute value of } \frac{a^2}{2}\begin{vmatrix} t_1(t_2-t_3) & (t_2-t_3) & 0 \\ t_3(t_2-t_1) & (t_2-t_1) & 0 \\ t_1t_3 & t_1+t_3 & 1 \end{vmatrix}$$

$$= \text{Absolute value of } \frac{a^2}{2}(t_2-t_3)(t_2-t_1)\begin{vmatrix} t_1 & 1 & 0 \\ t_3 & 1 & 0 \\ t_1t_3 & t_1+t_3 & 1 \end{vmatrix}$$

$$= \frac{a^2}{2}\,|(t_2-t_3)(t_2-t_1)(t_1-t_3)|$$

$$= \frac{1}{16a}\,|2a(t_1-t_2)\,2a(t_2-t_3)\,2a(t_3-t_1)$$

$$= \frac{1}{16a}\,|(y_1-y_2)(y_2-y_3)(y_3-y_1)| \text{ sq. units.}$$

21. Evaluate $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ dx on I⊂R / {–1, 1}.

Sol. Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\frac{2x}{1-x^2} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$$

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}(\tan 2\theta) = 2\theta + n\pi$$

Where $n = 0$ if $|x| < 1$

$= -1$ if $x > 1$

$= 1$ if $x < -1$

We have $d\theta = \frac{1}{1+x^2}\,dx$ and

$1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$

$$\therefore \int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$$

$$= \int \left(\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)(1 + x^2)\,\frac{1}{1+x^2}\,dx$$

$$= \int (2\theta + n\pi) \int \sec^2\theta\, d\theta$$

$$= 2\int \theta \sec^2\theta\, d\theta + n\pi) \int \sec^2\theta\, d\theta + c$$

$$= 2\,(\theta\tan\theta - \int \tan\theta\, d\theta) + n\pi\tan\theta + c$$

$$= 2\,(\theta\tan\theta + \log|\cos\theta| + n\pi\tan\theta + c$$

$$= (2\theta + n\pi)\tan\theta + 2\log\cos\theta + c$$

$$= (2\theta + n\pi)\tan\theta + \log\cos^2\theta + c$$

$$= (2\theta + n\pi)\tan\theta + \log\sec^2\theta + c$$

$$= x\tan^{-1}\left(\frac{2x}{1-x^2}\right) - \log(1 + x^2) + c$$

22. Evaluate $\int \sqrt{\frac{5-x}{x-2}}\,dx$ on (2, 5) (–1, 3).

Sol. $\int \sqrt{\frac{5-x}{x-2}} = \int \sqrt{\frac{(5-x)^2}{(x-2)(5-x)}}\,dx = \int \frac{5-x}{(x-2)(5-x)}\,dx$

$$= \int \frac{5-x}{\sqrt{5x-10-x^2+2x}}\,dx = \int \frac{5-x}{\sqrt{7x-10-x^2}}\,dx$$

Let $5 - x = A.\frac{d}{dx}(7x-10-x^2) + B \Rightarrow 5 - x = A(7-2x) + B$

Equating coeffs. of like terms

$-2A = -1 \Rightarrow A = \frac{1}{2}$

$7A + B = 5$

$7\left(+\frac{1}{2}\right) + B = 5 \Rightarrow B = 5 - \frac{7}{2} = \frac{3}{2}$

$\therefore 5 - x = \frac{1}{2}(7-2x) + \frac{3}{2}$

Then $\frac{5-x}{\sqrt{7x-10-x^2}} = \frac{1}{2}\frac{(7-2x)}{\sqrt{7x-10-x^2}} + \frac{\left(\frac{3}{2}\right)}{\sqrt{7x-10-x^2}}$

Now $\int \frac{5-x}{\sqrt{7x-10-x^2}}\,dx$

$$= \frac{1}{2}\int \frac{7-2x}{\sqrt{7x-10-x^2}}\,dx + \frac{3}{2}\int \frac{1}{\sqrt{7x-10-x^2}}\,dx$$

$$= \frac{1}{2}\cdot 2\sqrt{7x-10-x^2} + \frac{3}{2}\int \frac{1}{\sqrt{-x^2+7x-10}}\,dx$$

$\because -x^2 + 7x - 10$

$$= -[x^2 - 7x + 10] = -\left[\left(x-\frac{7}{2}\right)^2 + 10 - \frac{49}{4}\right] = -\left[\left(x-\frac{7}{2}\right)^2 - \frac{9}{4}\right]$$

$$= \frac{9}{4} - \left(x - \frac{7}{2}\right)^2 = \sqrt{7x - 10 - x^2} + \frac{3}{2}$$

$$\int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2 - \left(x - \frac{7}{2}\right)^2}} dx = \sqrt{7x - 10 - x^2} + \frac{3}{2} \sin^{-1}\left(\frac{x - \frac{7}{2}}{\frac{3}{2}}\right) + C$$

$$= \sqrt{7x - 10 - x^2} + \frac{3}{2} \sin^{-1}\left(\frac{2x - 7}{3}\right) + C$$

23. Evaluate $\int_5^7 \sqrt{\frac{7 - x}{x - 3}}\, dx.$

Sol. Put $x = 3\cos^2\theta + 7\sin^2\theta$

$dx = (7 - 3)\sin 2\theta\, d\theta$

$dx = 4\sin 2\theta\, d\theta$

U.L.

$x = 3\cos^2\theta + 7\sin^2\theta$

$7 = 3\cos^2\theta + 7\sin^2\theta$

$4\cos^2\theta = 0$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}$

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$x = 3\cos^2\theta + 7\sin^2\theta$

$3 = 3\sin^2\theta + 7\sin^2\theta$

$4\sin^2\theta = 0$

$\sin\theta = 0$

$\theta = 0$

$7 - x = 7 - (3\cos^2\theta + 7\sin^2\theta) = (7-3)\cos^2\theta = 4\cos^2\theta$

$x - 3 = 3\cos^2\theta + 7\sin^2\theta - 3 = (7-3)\sin^2\theta = 4\sin^2\theta$

$$\text{Let } I = \int_3^7 \sqrt{\frac{7-x}{x-3}}\, dx$$

$$I = \int_0^{\pi/2} \sqrt{\frac{4\cos^2\theta}{4\sin^2\theta}}\, 4(2)\sin\theta\cos\theta\, d\theta$$

$$= 8\int_0^{\pi/2} \cos^2\theta\, d\theta = 8\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{8}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{8}{2}\left[\frac{\pi}{2} + \frac{\sin 2\frac{\pi}{2}}{2}\right] - 0 = \frac{8}{2}\left[\frac{\pi}{2}\right] = 2\pi$$

24. Solve $\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - xy}$.

Sol. The given equation is a homogeneous equa-tion.

Put $y = vx$

$$\frac{dy}{dx} = v + x.\frac{dv}{dx},\ v + x.\frac{dv}{dx} = \frac{x^2(v^2 - 2v)}{x^2(1-v)}$$

$$x.\frac{dv}{dx} = \frac{v^2 - 2v}{1-v} - v = \frac{v^2 - 2v - v + v^2}{1-v} = \frac{2v^2 - 3v}{1-v}$$

$$\int \frac{1-v}{2v^2 - 3v}\, dv = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{3}\int\left(\frac{1}{v}+\frac{1}{2v-3}\right)dv = \log x - \log c$$

$$-\frac{1}{3}\left[\log v + \frac{1}{2}\log(2v-3)\right] = \log \frac{x}{c}$$

$$-\frac{1}{3}\log(v\sqrt{2v-3}) = \log \frac{x}{c}$$

$$\log v\sqrt{2v-3} = -3\log \frac{x}{c} = \log \frac{c^3}{x^3}$$

$$v\sqrt{2v-3} = \frac{c^3}{x^3}$$

$$x^3 v\sqrt{2v-3} = c^3$$

But $v = \frac{y}{x}$

$$x^3 . \frac{y}{x}\sqrt{\frac{2y}{x}-3} = c^3$$

$$x^2 y\sqrt{\frac{2y}{x}-3} = c^3$$

$xy\sqrt{2xy-3x^2} = c^3$ is the general solution.

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